

Propositional Calculus for Adjointness Lattices

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Abstract

Recently, Morsi has developed a complete syntax for the class of all adjointness algebras $(L; \cdot; A; K; H)$. There, $(L; \cdot)$ is a partially ordered set with top element 1, K is a conjunction on $(L; \cdot)$ for which 1 is a left identity element, and the two implication-like binary operations A and H on L are adjoints of K .

In this paper, we extend that formal system to one for the class ADJL of all 9-tuples $(L; \cdot; 1; 0; A; K; H; \wedge; _)$, called adjointness lattices; in each of which $(L; \cdot; 1; 0; \wedge; _)$ is a bounded lattice, and $(L; \cdot; A; K; H)$ is an adjointness algebra. We call it Propositional Calculus for Adjointness Lattices, abbreviated AdjLPC. Our axiom scheme for AdjLPC features four inference rules and thirteen axioms. We deduce enough theorems and inferences in AdjLPC to establish its completeness for ADJL; by means of a quotient-algebra structure (a Lindenbaum type of algebra). We study two negation-like unary operations in an adjointness lattice, defined by means of 0 together with A and H . We end by developing complete syntax for all adjointness lattices whose implications are S-type implications.

Keywords: Nonclassical logics; Syntax; Semantics; Adjointness; S-type implications

1 Propositional Calculus under Adjointness

In this section, we review the essentials of adjointness algebras, as well as the axioms, inference rules and main theorems of their complete syntax AdjPC [16]. We show how a complete syntax (with fewer axioms and inference rules) has been developed in [16] for a syntax EP-AdjPC; with the smaller semantical domain of all adjointness algebras whose implications satisfy the exchange principle.

1.1 Adjointness Algebras

The logic propositional calculus under adjointness, denoted by AdjPC, is based on partially ordered sets (posets) $(L; \cdot)$ whose elements are considered as truth values. Each poset is required to possess a top element 1; called truth or validity. The logic features three binary operations A; K and H on $(L; \cdot)$. The operation A, called an implication, should be antitone in the left argument and monotone in the right argument, and should have 1 as a left identity element; that is $A(1; z) = z \ \forall z \in L$. The operation K, called a conjunction, should be monotone in both arguments, and should also have 1 as a left identity element. (K need neither be commutative nor be associative, and may have no right identity element.) The operation H, called a forcing-implication, should be antitone in the left argument and monotone in the right argument, and should satisfy: $\forall y; z \in L: H(y; z) = 1 \iff y \cdot z$. (H need not have a left identity element.) The logic AdjPC offers complete syntax (a formal system for deriving theorems) for the semantical domain consisting of all the quintuples of the following definition.

Definition 1.1 [16] An adjointness algebra is a quintuple $(L; \cdot; A; K; H)$, in which $(L; \cdot)$ is a poset with a top element, A is an implication on $(L; \cdot)$, K is a conjunction on $(L; \cdot)$ and H is a forcing-implication on $(L; \cdot)$, subject to the condition that A; K and H are mutually related, for all $x; y; z$ in L, by

(Adjointness): $\forall x; y; z \in L: y \cdot A(x; z) \iff K(x; y) \cdot z \iff x \cdot H(y; z)$

We denote the class of all adjointness algebras by ADJ.

The subject of a possibly noncommutative, nonassociative conjunction K with two implication-like adjoints is an old one. See [2],[3],[4],[5],[6],[11],[19]. This idea lies also at the basis of that general trend in nonclassical logics collectively termed since 1990 substructural logics. Those are surveyed in the book [7], where one finds a detailed algebraic study of adjointness structures under the name residuated partially ordered groupoids, and a representation theorem for them is given in page 77 of [7]. See also Galois connections in [14]. The new contribution of [16] is the development of a complete syntax for those structures with weakest inference rules, in the general setting that 1 is a left identity, but not necessarily a right identity, for K.

The following are six basic inequalities in $(L; \cdot; A; K; H)$:

$$x \cdot H(A(x; z); z) \ , \ x \cdot H(y; K(x; y)) \ , \ y \cdot A(H(y; z); z) \ , \\ y \cdot A(x; K(x; y)) \ , \ K(x; A(x; z)) \cdot z \ , \ K(H(y; z); y) \cdot z$$

Lemma 1.1 [1] Let $\{x_j\}$ and $\{y_s\}$ be two subfamilies in an adjointness algebra $(L; \cdot; A; K; H)$ that have suprema in L , and let $\{z_t\}$ be a subfamily of L that has an infimum in L . Then

$$A \left(\sup_j x_j; \inf_t z_t \right) = \inf_{j;t} A(x_j; z_t), \quad (1)$$

$$K \left(\sup_j x_j; \sup_s y_s \right) = \sup_{j;s} K(x_j; y_s); \quad (2)$$

$$H \left(\sup_s y_s; \inf_t z_t \right) = \inf_{s;t} H(y_s; z_t); \quad (3)$$

Also, if $(L; \cdot)$ has a bottom element 0 , then $A(0; 0) = 1$ and $K(0; 1) = K(1; 0) = 0$:

Lemma 1.2 [18] Let $(L; \cdot)$ be a complete lattice. If an implication A on $(L; \cdot)$ satisfies (1), then there exist unique K and H such that $(L; \cdot; A; K; H)$ is an adjointness algebra. These are given by:

$$K(x; y) = \inf \{z \in L : y \cdot A(x; z)\}; \quad x; y \in L;$$

$$H(y; z) = \sup \{x \in L : y \cdot A(x; z)\}; \quad y; z \in L;$$

Similarly, a unique adjointness algebra will be obtained once a K on $(L; \cdot)$ that satisfies (2), or an H on $(L; \cdot)$ that satisfies (3), is given.

Lemma 1.2 demonstrates that adjointness algebras constitute a readily available framework for the study of implications and conjunctions related by adjointness. The special case that K is a supremum-preserving commutative triangular norm, and $A = H$ is its residuation implication, is well known.

1.2 Syntax: Axioms and Basic Theorems

The language of the propositional calculus under adjointness, AdjPC , features three logical connectives (binary operations) on the set WF of formulae: implication \rightarrow , conjunction $\&$ and forcing-implication \rightarrow_f . In an interpretation of AdjPC , the three logical connectives \rightarrow , $\&$ and \rightarrow_f will translate to the three operations A , K and H of some adjointness algebra, respectively. Lowercase Greek letters are used as metavariables running on formulae in WF .

Axioms of AdjPC [16]:

- P1: $\alpha \rightarrow_f (\alpha \rightarrow \beta) \rightarrow \beta$;
- P2: $\alpha \rightarrow_f ((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow \alpha$;
- P3: $((\alpha \rightarrow_f \beta) \rightarrow \beta) \rightarrow_f \alpha \rightarrow_f (\alpha \rightarrow_f \beta)$;
- P4: $\alpha \rightarrow_f (\alpha \rightarrow_f \beta) \rightarrow \beta$;
- P5: $(\alpha \rightarrow \beta) \rightarrow_f (\alpha \rightarrow_f \beta) \rightarrow \alpha$;
- P6: $((\alpha \rightarrow_f \beta) \rightarrow \beta) \rightarrow \alpha \rightarrow_f \beta$;
- P7: $\alpha \rightarrow_f \beta \rightarrow \alpha \rightarrow_f \beta$;

An inference $\alpha \rightarrow_f \beta$; $\alpha \rightarrow \beta$ is understood as usual; and is carried out by means of the four inference rules listed below. When $\alpha \rightarrow \beta$ (that is, β is derived from axioms alone), we write β , and we call β a theorem.

A considerable simplification of notation is achieved [16] by using a new symbol " $\frac{1}{2}\frac{3}{4}$ ". We write $\textcircled{\frac{1}{2}\frac{3}{4}}$ to abbreviate the writing of two formulae $\textcircled{\frac{3}{4}}$ and $\neg \textcircled{\frac{3}{4}}$. Thus, $\textcircled{\frac{1}{2}\frac{3}{4}}$ is a set of two formulae, and not one formula composed from two subformulae. So, an inference $\vdash \textcircled{\frac{1}{2}\frac{3}{4}}$ is, in fact, two inferences $\vdash \textcircled{\frac{3}{4}}$ and $\vdash \neg \textcircled{\frac{3}{4}}$. Likewise, a theorem $\vdash \textcircled{\frac{1}{2}\frac{3}{4}}$ is an abbreviated writing of two theorems. The meta-predicate $\vdash \textcircled{\frac{1}{2}\frac{3}{4}}$ is an equivalence relation on WF, called equality [16]. Another equivalence relation \sim on WF is defined by:

$$\textcircled{\frac{1}{2}\frac{3}{4}} \sim \textcircled{\frac{1}{2}\frac{3}{4}} \text{ iff } (\textcircled{\frac{1}{2}\frac{3}{4}} \text{ and } \neg \textcircled{\frac{1}{2}\frac{3}{4}}).$$

It follows from modus ponens, below, that if $\vdash \textcircled{\frac{1}{2}\frac{3}{4}}$ then $\textcircled{\frac{1}{2}\frac{3}{4}}$, but not vice-versa.

This logic is too general. It may be that no finite set of axioms can complete AdjPC if inference uses modus ponens (MP) alone! However, by adopting MP and three bits of the substitution theorem as inference rules, the seven axioms P1-P7 become complete for AdjPC.

Inference Rules of AdjPC [16]:

I1=MP: $\textcircled{\frac{1}{2}\frac{3}{4}}; \textcircled{\frac{3}{4}} \vdash \textcircled{\frac{1}{2}\frac{3}{4}}$ (Modus Ponens for forcing-implication).

I2: $\textcircled{\frac{1}{2}\frac{3}{4}} \vdash \textcircled{\frac{3}{4}} \text{ iff } (\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{1}{2}\frac{3}{4}})$ (substitution in left argument of $\frac{3}{4}$).

I3: $\textcircled{\frac{1}{2}\frac{3}{4}} \vdash \textcircled{\frac{3}{4}} \text{ iff } (\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{1}{2}\frac{3}{4}})$ (substitution in left argument of $\&$).

I4: $\textcircled{\frac{1}{2}\frac{3}{4}} \vdash \textcircled{\frac{3}{4}} \text{ iff } (\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{1}{2}\frac{3}{4}})$ (substitution in right argument of $\&$).

Proposition 1.1 [16] (transitivity of forcing-implication). $\textcircled{\frac{3}{4}}; \textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}$:

Theorem 1.1 [16] (reflexivity of forcing-implication). $\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}$.

The above two results establish that the meta-predicate $\textcircled{\frac{3}{4}}$ is a pre-order. In consequence, equality is an equivalence relation on WF.

Proposition 1.2 [16] The following are correct inferences in AdjPC:

$$\textcircled{\frac{3}{4}} \vdash \textcircled{\frac{3}{4}} \text{ iff } (\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}):$$

$$\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}} \vdash \textcircled{\frac{3}{4}} \text{ iff } (\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}):$$

$$\textcircled{\frac{3}{4}} \vdash \textcircled{\frac{3}{4}} \text{ iff } (\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}):$$

$$\textcircled{\frac{3}{4}} \text{ iff } (\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}):$$

$$\textcircled{\frac{3}{4}} \vdash \textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}:$$

$$\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}:$$

$$\textcircled{\frac{3}{4}} \text{ iff } (\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}):$$

$$\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}:$$

$$\textcircled{\frac{3}{4}}; \textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}:$$

$$\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}:$$

$$\textcircled{\frac{3}{4}}; \textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}:$$

$$\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}:$$

$$\textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}} \text{ iff } \textcircled{\frac{3}{4}}:$$

Theorem 1.2 [16] The following are theorems in AdjPC:

- $\vdash \textcircled{A} \supset (\neg \supset \textcircled{A} \& \neg)$;
- $\vdash (\neg \supset \textcircled{A} \circ) \& \neg \supset \textcircled{A} \circ$;
- $\vdash \textcircled{A} \& (\textcircled{A} \circ) \supset \textcircled{A} \circ$;
- $\vdash \neg \supset \textcircled{A} ((\neg \supset \textcircled{A} \circ)) \circ$;

Proposition 1.3 [16] (Adjointness): $\vdash \neg \supset \textcircled{A} (\textcircled{B} \circ) \circ \wedge \textcircled{B} \& \neg \supset \textcircled{A} \circ \wedge \textcircled{B} \supset \textcircled{A} (\neg \supset \textcircled{A} \circ)$:

This corresponds to the condition (Adjointness) in the definition of adjointness algebras (Definition 1.1).

AdjPC does not have a deduction theorem as strong as that of classical logic. For instance, the inference $\vdash \neg \supset \textcircled{A} \circ \wedge (\textcircled{B} \circ) \neg \supset \textcircled{A} (\textcircled{B} \circ) \circ$ is correct, but the formula $(\neg \supset \textcircled{A} \circ) \supset \textcircled{A} ((\textcircled{B} \circ) \neg \supset \textcircled{A} (\textcircled{B} \circ) \circ)$ is not a theorem in AdjPC. It is equivalent to the exchange principle for the implication \supset (Conclusion 1.2, below). However, the following important theorem holds in AdjPC.

Conclusion 1.1 [16]. Let \textcircled{A} be a theorem, and let \textcircled{B} be any formula in WF. Then $\vdash \textcircled{B} \supset \textcircled{A} \textcircled{B}$. In particular, \textcircled{A} and \textcircled{B} will be equivalent if and only if \textcircled{B} is also a theorem.

1.3 Semantics

An interpretation of AdjPC is [16] a pair $A = (L; \mathcal{V})$, in which $L = (L; \cdot; A; K; H)$ is an adjointness algebra, and $\mathcal{V} : WF \rightarrow L$ is called the valuation function of the interpretation; subject to the condition that the following three identities hold for all formulae $\textcircled{A}; \neg; \circ$:

- $\mathcal{V}(\textcircled{A} \circ) = A(\mathcal{V}(\textcircled{A}); \mathcal{V}(\circ))$;
- $\mathcal{V}(\textcircled{A} \& \neg) = K(\mathcal{V}(\textcircled{A}); \mathcal{V}(\neg))$;
- $\mathcal{V}(\neg \supset \textcircled{A} \circ) = H(\mathcal{V}(\neg); \mathcal{V}(\circ))$;

If $\mathcal{V}(\textcircled{A}) = 1$, we say that \textcircled{A} is true in A , and we write $A \models \textcircled{A}$. Given a set \mathcal{J} in WF, if $A \models \textcircled{A}$ for all $\mathcal{J} \in \mathcal{J}$, we write $A \models \mathcal{J}$. If $A \models \mathcal{J}$ for every interpretation A such that $A \models \mathcal{J}$, we write $\mathcal{J} \models \pm$. If $A \models \textcircled{A}$ for all interpretations A , we say \textcircled{A} is universally valid (or, a tautology), and we write $\models \textcircled{A}$.

Semantics-Theorem 1.1[16] Suppose $\mathcal{J} \models \textcircled{A}$ for some set of formulae $\mathcal{J} \in \text{f}^{\textcircled{A}}$. Then $\mathcal{J} \models \textcircled{A}$. Consequently, AdjPC is sound for its semantics, in the sense that if $\vdash \textcircled{A}$ then $\models \textcircled{A}$.

Semantics-Theorem 1.2[16] AdjPC is complete for ADJ; in the sense that its theorems are its universally valid formulae.

Corollary 1.1 [16] Two formulae $\textcircled{A}; \neg$ will be equivalent if and only if $\mathcal{V}(\textcircled{A}) = \mathcal{V}(\neg)$ in all interpretations $(L; \mathcal{V})$.

The converse of Semantics-Theorem 1.1 fails. For instance, we have $\models \circ \supset \neg \supset \textcircled{A} \circ$, but the inference $\vdash \circ \supset \neg \supset \textcircled{A} \circ$ is incorrect [16]. However, that converse holds if all formulae in \mathcal{J} are equational; that is, they take the form $\neg \supset \textcircled{A} \circ$ [16].

1.4 The Exchange Principle

An implication A is said to satisfy the exchange principle [22] if it satisfies:

$$\text{EP: } \quad \forall x, y, z \in L : \quad A(x; A(y; z)) = A(y; A(x; z)) :$$

Morsi [16] has developed a complete syntax for the smaller semantical domain EP-ADJ of all adjointness algebras whose implications satisfy EP. He called it propositional calculus under adjointness and exchange principle, denoted EP-AdjPC. Its language is the same as that of AdjPC. A preliminary choice of the axioms of EP-AdjPC would be to augment the seven axioms of AdjPC with:

$$\text{EP: } \quad (\otimes) \quad (\neg) \quad (\circ) \quad \frac{3}{4} \quad (\neg) \quad (\otimes) \quad (\circ) :$$

And we may retain I1-I4 as four inference rules for EP-AdjPC. Then EP-AdjPC would become sound and complete for EP-ADJ. However, it is possible to extract a smaller axiom scheme for EP-AdjPC; in the manner shown below.

Conclusion 1.2 [16]. The following eight schema of equational formulae are equivalent in AdjPC:

$$\begin{aligned} \text{E1=EP: } & \quad (\otimes) \quad (\neg) \quad (\circ) \quad \frac{3}{4} \quad (\neg) \quad (\otimes) \quad (\circ) \quad \text{(exchange principle for } \otimes \text{)} . \\ \text{E2: } & \quad _ \& (\otimes) \quad (\circ) \quad \frac{3}{4} \quad (\otimes) \quad _ \& (\circ) : \\ \text{E3: } & \quad \otimes \& (_ \& \circ) \quad \frac{3}{4} \quad _ \& (\otimes \& \circ) \quad \text{(exchange principle for } \& \text{)} . \\ \text{E4: } & \quad (\neg \frac{3}{4} \circ) \quad \frac{3}{4} \quad (\otimes \& \neg \frac{3}{4} \otimes \& \circ) : \\ \text{E5: } & \quad (\neg \frac{3}{4} (\otimes) \circ) \quad \frac{3}{4} \quad (\otimes \& \neg \frac{3}{4} \circ) : \\ \text{E6: } & \quad (\pm \frac{3}{4} \circ) \& \neg \frac{3}{4} ((\neg \frac{3}{4} \pm) \circ) : \\ \text{E7: } & \quad (\otimes \& \neg \frac{3}{4} \circ) \quad \frac{3}{4} \quad (\neg \frac{3}{4} (\otimes) \circ) : \\ \text{E8: } & \quad (\neg \frac{3}{4} \circ) \quad \frac{3}{4} \quad ((\otimes) \neg) \quad \frac{3}{4} \quad (\otimes) \quad (\circ) : \end{aligned}$$

It follows from this conclusion that P1-P7 and E1-E8 are theorems of EP-AdjPC. A new axiom scheme for EP-AdjPC has been chosen from among these ...fteen theorems.

Axioms of EP-AdjPC [16]:

$$\begin{aligned} \text{P1: } & \quad \circ \quad \frac{3}{4} \quad (\otimes) \quad (\circ) : \\ \text{P2: } & \quad \otimes \quad \frac{3}{4} \quad ((\otimes) \quad (\circ) \quad \frac{3}{4} \quad \circ) : \\ \text{P3: } & \quad (((\otimes \frac{3}{4} \neg) \quad \neg) \quad \frac{3}{4} \quad \circ) \quad \frac{3}{4} \quad (\otimes \frac{3}{4} \circ) : \\ \text{P6: } & \quad ((\neg \frac{3}{4} \circ) \& \neg) \& \pm \frac{3}{4} \circ \& \pm : \\ \text{E5: } & \quad (\neg \frac{3}{4} (\otimes) \circ) \quad \frac{3}{4} \quad (\otimes \& \neg \frac{3}{4} \circ) : \\ \text{E7: } & \quad (\otimes \& \neg \frac{3}{4} \circ) \quad \frac{3}{4} \quad (\neg \frac{3}{4} (\otimes) \circ) : \end{aligned}$$

Inference Rules of EP-AdjPC [16]: MP, I2 and I3.

The formal system built upon these six axioms and three inference rules is sound and complete for EP-ADJ [16].

2 Adjointness Lattices

Definition 2.1 An adjointness lattice is a 9-tuple $(L; \cdot; 1; 0; A; K; H; \wedge; _)$, in which $(L; \cdot; 1; 0; \wedge; _)$ is a bounded lattice, and $(L; \cdot; A; K; H)$ is an adjointness algebra. We denote the class of all adjointness lattices by ADJL.

We aim to develop a complete syntax for the semantical domain ADJL. We call it propositional calculus for adjointness lattices, and denote it by AdjLPC. We select the axioms for AdjLPC from among the many inequalities derived algebraically in ADJL. Since the logic AdjLPC is an extension of AdjPC, the seven axioms of AdjPC can be adopted, and we choose six new axioms, namely, the following universally valid inequalities in ADJL:

- M8: $x \wedge y \cdot x _ z$:
M9: $x _ x \cdot x$:
M10: $H(y; z) \cdot H(x _ y; z _ x)$:
M11: $x \cdot x \wedge x$:
M12: $H(y; z) \cdot H(x \wedge y; z \wedge x)$:
M13: $0 \cdot x$:

In forms free from \cdot , these relations become: for all $x; y; z$ in L:

- N8: $H(x \wedge y; x _ z) = 1$:
N9: $H(x _ x; x) = 1$:
N10: $H(H(y; z); H(x _ y; z _ x)) = 1$:
N11: $H(x; x \wedge x) = 1$:
N12: $H(H(y; z); H(x \wedge y; z \wedge x)) = 1$:
N13: $H(0; x) = 1$:

3 Syntax: Language, Axioms and Inference Rules

The language of the Propositional Calculus for Adjointness Lattices, AdjLPC, consists of a denumerable set WF of formulae and ...ve logical connectives (binary operations) on WF: implication \rightarrow , conjunction $\&$, forcing-implication $\overset{\circ}{\rightarrow}$, weak conjunction \wedge and disjunction $_$. The set WF is constructed from a denumerable subset WF_0 of atomic formulae by means of repeated application of the logical connectives. We also add to WF_0 a special element \perp called Falsum. We denote P1 by \top (Truth). As usual, brackets and comma are secondary symbols in the language.

To reduce the number of brackets appearing in complex formulae, we maintain a convention of priority among the eight operation symbols $\rightarrow; \&; \overset{\circ}{\rightarrow}; \wedge; _; \perp; \top$. We give $\&; \wedge; _$ the highest priority; whereas we give $\perp; \top$ lower priority than the other symbols.

In Section 2, we identified six identities N8-N13 (equivalently, six inequalities M8-M13) valid in all adjointness lattices. Their corresponding statements on formulae, together with the seven axioms P1-P7 of AdjPC, are what follows:

Axioms of AdjLPC: The following are theorems:

- P1: $\perp \overset{\circ}{\rightarrow} (\top) \perp$:
P2: $\top \overset{\circ}{\rightarrow} ((\top) \perp) \overset{\circ}{\rightarrow} \perp$:
P3: $((\top \overset{\circ}{\rightarrow} \perp) \perp) \overset{\circ}{\rightarrow} \perp \overset{\circ}{\rightarrow} (\top \overset{\circ}{\rightarrow} \perp)$:
P4: $\perp \overset{\circ}{\rightarrow} (\top) \& \perp$:
P5: $(\top) (\perp \overset{\circ}{\rightarrow} \perp) \& \perp \overset{\circ}{\rightarrow} (\top)$:
P6: $((\perp \overset{\circ}{\rightarrow} \perp) \& \perp) \& \perp \overset{\circ}{\rightarrow} \perp$:
P7: $\top \& \perp \overset{\circ}{\rightarrow} \perp$:

- P8: $\text{A} \wedge \text{B} \rightarrow \text{A} \rightarrow \text{B}$;
P9: $\text{A} \rightarrow \text{B} \rightarrow \text{A} \rightarrow \text{C} \rightarrow \text{B} \rightarrow \text{C}$;
P10: $(\text{A} \rightarrow \text{B}) \rightarrow (\text{A} \rightarrow \text{C}) \rightarrow (\text{A} \rightarrow \text{B} \wedge \text{C})$;
P11: $\text{A} \rightarrow \text{B} \rightarrow \text{A} \rightarrow \text{C} \rightarrow \text{B} \rightarrow \text{C}$;
P12: $(\text{A} \rightarrow \text{B}) \rightarrow (\text{A} \rightarrow \text{C}) \rightarrow (\text{A} \rightarrow \text{B} \wedge \text{C})$;
P13: $\text{A} \rightarrow \text{B} \rightarrow \text{A} \rightarrow \text{C} \rightarrow \text{B} \rightarrow \text{C}$;

Inference Rules for AdjLPC are those of AdjPC: MP, I2, I3 and I4.

In an interpretation of AdjLPC, the ...ve logical connectives \rightarrow ; \wedge ; \vee ; \neg and \perp will translate onto the ...ve operations A; K; H; \wedge and \vee of some adjointness lattice, respectively, whereas Falsum \perp will translate onto 0. Also, formulae will translate onto functions on truth values; built up as composites of A; K; H; \wedge ; \vee and 0. AdjLPC will be sound for these semantics, in the sense that all theorems will translate onto functions that are identically equal to 1.

4 Syntax: Essential Theorems

We derive enough theorems and inferences (called propositions) in AdjLPC to establish, in Section 5, its completeness for the semantical domain ADJL of adjointness lattices. In most proofs we shall use, as matters of course, both MP and the reflexivity and transitivity of the binary relation \rightarrow .

Theorem 4.1 $\text{A} \rightarrow \text{B} \rightarrow \text{A} \rightarrow \text{C} \rightarrow \text{B} \rightarrow \text{C}$ and $\text{A} \wedge \text{B} \rightarrow \text{A} \rightarrow \text{B}$:

Proof. Use P8 with $\text{A} \rightarrow \text{B}$, then P11 to derive $\text{A} \rightarrow \text{B} \rightarrow \text{A} \rightarrow \text{C} \rightarrow \text{B} \rightarrow \text{C}$. The other part follows similarly. ■

Theorem 4.2 (idempotent laws for disjunction and for weak conjunction).
 $\text{A} \vee \text{A} \rightarrow \text{A}$ and $\text{A} \wedge \text{A} \rightarrow \text{A}$:

Proof. Apply Theorem 4.1, P9 and P11. ■

Theorem 4.3 (commutative laws for disjunction and for weak conjunction).
 $\text{A} \vee \text{B} \rightarrow \text{B} \vee \text{A}$ and $\text{A} \wedge \text{B} \rightarrow \text{B} \wedge \text{A}$:

Proof. By P10, $\text{A} \rightarrow (\text{B} \vee \text{A}) \rightarrow (\text{A} \vee \text{B})$. So, by $\text{A} \rightarrow \text{B} \rightarrow \text{A} \rightarrow \text{C} \rightarrow \text{B} \rightarrow \text{C}$ and MP we infer the ...rst part. The second part follows similarly. ■

Theorem 4.4 $(\text{A} \rightarrow \text{B}) \rightarrow (\text{A} \rightarrow \text{C}) \rightarrow (\text{A} \rightarrow \text{B} \wedge \text{C})$; $(\text{A} \rightarrow \text{B}) \rightarrow (\text{A} \rightarrow \text{C}) \rightarrow (\text{A} \rightarrow \text{B} \vee \text{C})$,
 $(\text{A} \rightarrow \text{B}) \rightarrow (\text{A} \rightarrow \text{C}) \rightarrow (\text{A} \rightarrow \text{B} \wedge \text{C})$, $(\text{A} \rightarrow \text{B}) \rightarrow (\text{A} \rightarrow \text{C}) \rightarrow (\text{A} \rightarrow \text{B} \vee \text{C})$:

Proof. These follow clearly from P10, P12 and commutivity (Theorem 4.3). ■

Applying MP on the preceding theorem, we obtain

Proposition 4.1 (monotonicity). $\text{A} \rightarrow \text{B} \rightarrow \text{A} \rightarrow \text{C} \rightarrow \text{B} \rightarrow \text{C}$;
 $\text{A} \rightarrow \text{B} \rightarrow \text{A} \rightarrow \text{C} \rightarrow \text{B} \rightarrow \text{C}$, $\text{A} \rightarrow \text{B} \rightarrow \text{A} \rightarrow \text{C} \rightarrow \text{B} \rightarrow \text{C}$, $\text{A} \rightarrow \text{B} \rightarrow \text{A} \rightarrow \text{C} \rightarrow \text{B} \rightarrow \text{C}$:

Proposition 4.2 (Substitution Theorem). $\vdash \text{a}(\text{b}) \iff \text{a}(\text{b}^j)$:

Where $\text{a}(\text{b})$ is a formula in which b occurs as a subformula, and $\text{a}(\text{b}^j)$ is a formula obtained from $\text{a}(\text{b})$ by substituting b^j for b , in one or more of the occurrences of b of in $\text{a}(\text{b})$. In particular, substitution preserves equivalidity.

Proof. This follows clearly from all the monotonicity propositions of the ...ve logical connectives \neg , \wedge and $_$. ■

Theorem 4.5 $\vdash (\text{a} \supset \text{b}) \iff (\text{a} \supset \text{b}^j)$:

- Proof.** (1) $(\text{a} \supset \text{b}) \supset (\text{a} \supset \text{b}^j)$ (Theorem 4.4)
 (2) $(\text{a} \supset \text{b}^j) \supset (\text{a} \supset \text{b})$ ((1), Theorem 4.2, Substitution Theorem)
 (3) $\text{a} \supset \text{b} \iff \text{a} \supset \text{b}^j$ (Theorem 4.1)
 (4) $(\text{a} \supset \text{b}^j) \iff (\text{a} \supset \text{b})$ ((3), Proposition 1.2). ■

Theorem 4.6 $\vdash (\text{a} \wedge \text{b}) \iff (\text{a} \wedge \text{b}^j)$:

Proof. Similar. ■

Proposition 4.3 $\vdash \text{a} \wedge \text{b} \iff \text{a} \wedge \text{b}^j$:

Proof. Follows by Theorem 4.5 and by Theorem 4.1. ■

Proposition 4.4 $\vdash \text{a} \supset \text{b} \iff \text{a} \supset \text{b}^j$:

- Proof.** (1) $\text{a} \supset \text{b} \supset \text{a} \supset \text{b}^j$ (...rst hypothesis, Proposition 4.1)
 (2) $\text{a} \supset \text{b}^j \supset \text{a} \supset \text{b}$ (second hypothesis, Proposition 4.3)
 (3) $\text{a} \supset \text{b} \iff \text{a} \supset \text{b}^j$ ((1), (2)).

The opposite inference follows from Theorem 4.1. ■

Proposition 4.5 $\vdash \text{a} \supset \text{b} \iff \text{a} \supset \text{b}^j$:

Proof. Similar. ■

Theorem 4.7 $\vdash \text{a} \wedge \text{b} \iff \text{a} \wedge \text{b}^j$ and $\vdash \text{a} \supset \text{b} \iff \text{a} \supset \text{b}^j$:

Proof. Use Theorem 4.1 together with Proposition 4.3. ■

5 Semantics

We explain how ADJL (cf. Section 2) constitutes a semantical domain for AdjLPC. We prove that the syntax of AdjLPC is sound for ADJL. We then show that the quotient of the tuple $(WF; \text{ } \vdash; \text{ } \supset; \text{ } \wedge; \text{ } _)$, with respect to the relation equivalidity, is a model of AdjLPC. We use it to prove completeness.

An interpretation of AdjLPC is a pair $T = (L; \text{ } \vdash)$, in which

$$L = (L; \text{ } \vdash; \text{ } \supset; \text{ } \wedge; \text{ } _)$$

is an adjointness lattice, and \mathcal{V} is a function from the set WF of formulae into L, called the valuation function (or truth function) of the interpretation, subject to the condition that the following six identities hold for all formulae $\mathbb{A}, \mathbb{B}, \mathbb{C}$:

$$\mathcal{V}(\mathbb{A} \supset \mathbb{B}) = A(\mathcal{V}(\mathbb{A}); \mathcal{V}(\mathbb{B})); \quad (4)$$

$$\mathcal{V}(\mathbb{A} \& \mathbb{B}) = K(\mathcal{V}(\mathbb{A}); \mathcal{V}(\mathbb{B})); \quad (5)$$

$$\mathcal{V}(\neg \mathbb{A}) = H(\mathcal{V}(\mathbb{A})); \quad (6)$$

$$\mathcal{V}(\mathbb{A} \wedge \mathbb{B}) = \mathcal{V}(\mathbb{A}) \wedge \mathcal{V}(\mathbb{B}); \quad (7)$$

$$\mathcal{V}(\mathbb{A} _ \mathbb{B}) = \mathcal{V}(\mathbb{A}) _ \mathcal{V}(\mathbb{B}); \quad (8)$$

$$\mathcal{V}(\perp) = 0; \quad (9)$$

$\mathcal{V}(\mathbb{A}) \in L$ (also denoted by \mathbb{A}) is called the validity (or, truth) of \mathbb{A} in this interpretation. The symbol \mathbb{A} is understood as in AdjPC (Subsection 1.3).

Semantics-Theorem 5.1. AdjLPC is sound for its semantics, in the sense that if $\vdash \mathbb{A}$ then \mathbb{A} ; that is, all theorems are universally valid.

Proof. By the identities N8-N13, we know that the axioms P8-P13 are universally valid in ADJL. Also, AdjLPC has the same inference rules as AdjPC. We can therefore imitate the proof in [16] of Semantics-Theorem 1.1, and deduce that AdjLPC is sound for ADJL. ■

We next address the question of the completeness of AdjLPC for ADJL. We follow a standard procedure due to Lindenbaum and Tarski. We begin by constructing the natural interpretation of AdjLPC. Denote the equivalence relation equivalidity (on WF) simply by \sim . Let $p : WF \rightarrow WF/\sim$ be the quotient map. Then a partial order \cdot is well-defined on WF/\sim by: $\mathbb{A} \cdot \mathbb{B}$ iff $\mathbb{A} \supset \mathbb{B}$. By imitating for AdjLPC the proof in [16] of Conclusion 1.1, we find that the poset $(WF/\sim; \cdot)$ has a top element which is precisely the set of all theorems in AdjLPC. We denote this top element by 1. Also by P13, the poset $(WF/\sim; \cdot)$ has a bottom element which is precisely the equivalence class of Falsum \perp . We denote this bottom element by 0. Moreover, the Substitution Theorem guarantees that the above logical connectives $\supset; \&; \neg; \wedge$ and $_$ possess the substitution property for \sim . In consequence, the following above binary operations $A; K; H; \wedge; _$ are well defined on WF/\sim . For all $\mathbb{A}; \mathbb{B}; \mathbb{C}$ in WF/\sim :

$$A(\mathbb{A}; \mathbb{B}) = p(\mathbb{A} \supset \mathbb{B}),$$

$$K(\mathbb{A}; \mathbb{B}) = p(\mathbb{A} \& \mathbb{B}),$$

$$H(\mathbb{A}) = p(\neg \mathbb{A}),$$

$$\mathbb{A} \wedge \mathbb{B} = p(\mathbb{A} \wedge \mathbb{B}),$$

$$\mathbb{A} _ \mathbb{B} = p(\mathbb{A} _ \mathbb{B}).$$

From [16], we know that $(WF/\sim; \cdot; A; K; H)$ is an adjointness algebra. Hence, we need only prove that $(WF/\sim; \cdot; 1; 0; \wedge; _)$ is a bounded lattice. But, this follows in a routine way from the axioms and from the theorems and propositions of Section 4. This completes the proof that $\mathcal{L} = (WF/\sim; \cdot; 1; 0; \wedge; _)$ is an adjointness lattice. Finally, by their construction,

$\mathcal{A}; \mathcal{K}; \mathcal{H}; \mathcal{A}; \leq; 0$ and p satisfy the conditions (4)-(9) for p to become a valuation function. This demonstrates that the pair $\mathcal{L}; p$ is an interpretation of AdjLPC. It is called the natural interpretation of AdjLPC.

Since, for any formula ϕ , we have $\mathcal{L}; p \models \phi$ (that is, ϕ is the top element of $(WF, \leq; \cdot)$) if and only if ϕ is a theorem, then in the light of Semantics-Theorem 5.1 we obtain:

Semantics-Theorem 5.2. AdjLPC is complete for ADLJ; in the sense that its theorems are its universally valid formulae (that is, ϕ if and only if $\models \phi$, for all formulae ϕ).

Semantics-Theorem 5.3. Let \mathcal{J} be a nonempty set of equational formulae. Then for any formula ϕ in WF , if and only if $\mathcal{J} \vdash \phi$ if and only if $\mathcal{J} \models \phi$.

Proof. Adjoin \mathcal{J} to the set of axioms, then repeat all the arguments above.

■

Along the same lines of Subsection 1.4, we also possess a complete syntax for the semantical domain EP-ADJL of all adjointness lattices whose implications satisfy EP. We call it propositional calculus for adjointness lattices and exchange principle, and we denote it by EP-AdjLPC. The language of EP-AdjLPC is the same as that of AdjLPC. Our axiom scheme for EP-AdjLPC features three inference rules and twelve axioms. The inference rules and the first six axioms are those of EP-AdjPC, whereas the last six axioms P8-P13 are as in AdjLPC. From Subsection 1.4 and this section, EP-AdjLPC is sound and complete for EP-ADJL.

6 Syntax: Additional Theorems

In this section we prove further useful theorems and inferences in AdjLPC.

Theorem 6.1 $\vdash \phi \& \psi \rightarrow \psi \& \phi$?:

Proof. By P7 and P13. ■

Theorem 6.2 $\vdash \phi \& \psi \rightarrow \psi \& \phi$?:

Proof. By P13, $\vdash \phi \rightarrow \psi \rightarrow \psi \& \phi$ (Adjointness), $\vdash \psi \& \phi \rightarrow \psi \& \phi$. This and P13 yield the stated equivalency. ■

Theorem 6.3 $\vdash \phi \wedge \psi \rightarrow \psi \wedge \phi$? , $\vdash \phi \rightarrow \psi \rightarrow \psi \wedge \phi$? , $\vdash \phi \rightarrow \psi \rightarrow \psi \wedge \phi$? , $\vdash \phi \rightarrow \psi \rightarrow \psi \wedge \phi$?:

Proof. These follow clearly from P13 and Proposition 4.3. ■

Theorem 6.4 $\vdash \phi \rightarrow \psi \rightarrow \psi \wedge \phi$?: In particular, $\vdash \phi \rightarrow \psi \rightarrow \psi \wedge \phi$?.

Proof. By P13, $\vdash \phi \rightarrow \psi \rightarrow \psi \wedge \phi$ ($\vdash \phi \rightarrow \psi \rightarrow \psi \wedge \phi$), from which we get by (Adjointness), $\vdash \phi \rightarrow \psi \rightarrow \psi \wedge \phi$ ($\vdash \phi \rightarrow \psi \rightarrow \psi \wedge \phi$). So by MP, $\vdash \phi \rightarrow \psi \rightarrow \psi \wedge \phi$: ■

Theorem 6.5 $\vdash (\phi \rightarrow \psi) \rightarrow (\psi \rightarrow \phi) \rightarrow \phi \rightarrow \psi$?:

Proof. That $\neg (\mathbb{R})^- \wedge \circ \frac{3}{4} (\mathbb{R})^- \wedge (\mathbb{R})^\circ$ follows from Theorem 4.1 and Propositions 1.2, 4.4. The other half is proved as follows:

- (1) $(\mathbb{R})^- \wedge (\mathbb{R})^\circ \frac{3}{4} (\mathbb{R})^-$ (Theorem 4.1)
- (2) $(\mathbb{R})^- \wedge (\mathbb{R})^\circ \frac{3}{4} (\mathbb{R})^\circ$ (Theorem 4.1)
- (3) $\mathbb{R} \& ((\mathbb{R})^- \wedge (\mathbb{R})^\circ) \frac{3}{4}^-$ ((1), Adjointness)
- (4) $\mathbb{R} \& ((\mathbb{R})^- \wedge (\mathbb{R})^\circ) \frac{3}{4}^\circ$ ((2), Adjointness)
- (5) $\mathbb{R} \& ((\mathbb{R})^- \wedge (\mathbb{R})^\circ) \frac{3}{4}^- \wedge \circ$ ((3), (4), Proposition 4.4)
- (6) $(\mathbb{R})^- \wedge (\mathbb{R})^\circ \frac{3}{4} (\mathbb{R})^- \wedge \circ$ ((5), Adjointness). ■

Theorem 6.6 $\neg (\mathbb{R}^-)^\circ \frac{1}{2} \frac{3}{4} (\mathbb{R})^\circ \wedge (\neg)^\circ$:

Proof. That $\neg (\mathbb{R}^-)^\circ \frac{3}{4} (\mathbb{R})^\circ \wedge (\neg)^\circ$ follows from Theorem 4.1 and Propositions 1.2, 4.4. The other half is proved as follows:

- (1) $(\mathbb{R})^\circ \wedge (\neg)^\circ \frac{3}{4} (\mathbb{R})^\circ$ (Theorem 4.1)
- (2) $(\mathbb{R})^\circ \wedge (\neg)^\circ \frac{3}{4} (\neg)^\circ$ (Theorem 4.1)
- (3) $\mathbb{R} \frac{3}{4} ((\mathbb{R})^\circ \wedge (\neg)^\circ) \frac{3}{4}^\circ$ ((1), Adjointness)
- (4) $^- \frac{3}{4} ((\mathbb{R})^\circ \wedge (\neg)^\circ) \frac{3}{4}^\circ$ ((2), Adjointness)
- (5) $\mathbb{R}^- \frac{3}{4} ((\mathbb{R})^\circ \wedge (\neg)^\circ) \frac{3}{4}^\circ$ ((3), (4), Proposition 4.5)
- (6) $(\mathbb{R})^\circ \wedge (\neg)^\circ \frac{3}{4} (\mathbb{R}^-)^\circ$ ((5), Adjointness). ■

The next two theorems have proofs along lines similar to the above two.

Theorem 6.7 $\neg (\mathbb{R} \frac{3}{4}^- \wedge \circ) \frac{1}{2} \frac{3}{4} (\mathbb{R} \frac{3}{4}^-) \wedge (\mathbb{R} \frac{3}{4}^\circ)$:

Theorem 6.8 $\neg (\mathbb{R}^- \frac{3}{4}^\circ) \frac{1}{2} \frac{3}{4} (\mathbb{R} \frac{3}{4}^\circ) \wedge (\neg \frac{3}{4}^\circ)$:

Theorem 6.9 $\neg \mathbb{R} \& (\neg^-)^\circ \frac{1}{2} \frac{3}{4} (\mathbb{R} \&^-) _ (\mathbb{R} \&^\circ)$:

Proof. That $(\mathbb{R} \&^-) _ (\mathbb{R} \&^\circ) \frac{3}{4} \mathbb{R} \& (\neg^-)^\circ$ follows from Theorem 4.1 and Propositions 1.2, 4.5. The other half is proved as follows:

- (1) $\mathbb{R} \&^- \frac{3}{4} (\mathbb{R} \&^-) _ (\mathbb{R} \&^\circ)$ (Theorem 4.1)
- (2) $\mathbb{R} \&^\circ \frac{3}{4} (\mathbb{R} \&^-) _ (\mathbb{R} \&^\circ)$ (Theorem 4.1)
- (3) $^- \frac{3}{4} (\mathbb{R}) _ (\mathbb{R} \&^-) _ (\mathbb{R} \&^\circ)$ ((1), Adjointness)
- (4) $^\circ \frac{3}{4} (\mathbb{R}) _ (\mathbb{R} \&^-) _ (\mathbb{R} \&^\circ)$ ((2), Adjointness)
- (5) $^- _ \frac{3}{4} (\mathbb{R}) _ (\mathbb{R} \&^-) _ (\mathbb{R} \&^\circ)$ ((3), (4), Proposition 4.5)
- (6) $\mathbb{R} \& (\neg^-)^\circ \frac{3}{4} (\mathbb{R} \&^-) _ (\mathbb{R} \&^\circ)$ ((5), Adjointness). ■

Theorem 6.10 $\neg (\mathbb{R}^-) \&^\circ \frac{1}{2} \frac{3}{4} (\mathbb{R} \&^\circ) _ (\neg \&^\circ)$:

Proof. Similar. ■

Proposition 6.1 $(\mathbb{R}^-) _ \frac{3}{4}^\circ \frown \mathbb{R} \frac{3}{4}^\circ; \neg \frac{3}{4}^\circ; _ \frac{3}{4}^\circ \mathbb{g} \frown (\mathbb{R}^-) _ \frac{3}{4}^\circ$:

Proof. These equivalences follow easily from Proposition 4.5. ■

Proposition 6.2 $^\circ \frac{3}{4} (\mathbb{R} \wedge \neg) \wedge _ \frown \mathbb{R} \frac{3}{4} \mathbb{R}; \circ \frac{3}{4}^-; \circ \frac{3}{4} \mathbb{g} \frown \circ \frac{3}{4} \mathbb{R} \wedge (\neg \wedge _)$:

Proof. These equivalences follow easily from Proposition 4.4. ■

Theorem 6.11 $\neg (\mathbb{R} \rightarrow \neg) \equiv \frac{1}{2} \mathbb{R} \rightarrow (\neg \rightarrow)$:

Proof. Apply Proposition 6.1 twice, with $\frac{\mathbb{R}(\neg \rightarrow)}{\circ}$ and with $\frac{\mathbb{R}(\rightarrow)}{\circ}$. ■

Theorem 6.12 $\neg (\mathbb{R} \wedge \neg) \equiv \frac{1}{2} \mathbb{R} \wedge (\neg \wedge)$:

Proof. Similar, using Proposition 6.2. ■

Using the monotonicity properties of \rightarrow ; $\frac{3}{4}$; $\&$ (Proposition 1.2), it is easy to conclude

Theorem 6.13 $\neg (\mathbb{R}) \rightarrow (\mathbb{R}) \equiv \frac{3}{4} (\mathbb{R}) \rightarrow \rightarrow$;

$\neg (\mathbb{R}) \rightarrow (\neg) \equiv \frac{3}{4} (\mathbb{R} \wedge \neg) \rightarrow$;

$\neg (\mathbb{R} \frac{3}{4} \neg) \equiv (\mathbb{R} \frac{3}{4} \rightarrow) \frac{3}{4} (\mathbb{R} \frac{3}{4} \neg \rightarrow)$;

$\neg (\mathbb{R} \frac{3}{4} \rightarrow) \equiv (\neg \frac{3}{4} \rightarrow) \frac{3}{4} (\mathbb{R} \wedge \neg \frac{3}{4} \rightarrow)$;

$\neg (\mathbb{R} \wedge \neg) \& \equiv \frac{3}{4} (\mathbb{R} \& \rightarrow) \wedge (\neg \& \rightarrow)$;

$\neg \mathbb{R} \& (\neg \wedge \rightarrow) \equiv \frac{3}{4} (\mathbb{R} \& \neg) \wedge (\mathbb{R} \& \rightarrow)$;

Theorem 6.14 $\neg (>) \equiv \mathbb{R} \frac{1}{2} \mathbb{R} \rightarrow$ and $\neg \mathbb{R} >$.

Proof. Use Proposition 1.2, $>$ and MP. ■

7 Negations from Implications

A negation n on $(L; \cdot, 1; 0)$ is an order-reversing map that satisfies, $n(0) = 1$ and $n(1) = 0$, and it is a strong negation if it is also an involution; that is, $n(n(x)) = x$ for all x [21].

In an adjointness lattice, we define two functions $n, m : L \rightarrow L$ by:

$$n(x) = A(x; 0); \tag{10}$$

$$m(y) = H(y; 0); \tag{11}$$

It is easy to see that n is a negation on $(L; \cdot)$, whereas m may lack the property $m(1) = 0$. In the syntax of AdjLPC, the corresponding two unary operations $\neg; \#$ on WF are defined on a formula \mathbb{R} by:

$\neg \mathbb{R} = \mathbb{R} \rightarrow ?$, $\# \mathbb{R} = \neg \frac{3}{4} ?$;

We have the following properties for \neg and $\#$.

Proposition 7.1 $\mathbb{R} \frac{3}{4} \neg \equiv \neg \frac{3}{4} \mathbb{R}$ and $\mathbb{R} \frac{3}{4} \neg \equiv \# \frac{3}{4} \# \mathbb{R}$;

Proof. Use Proposition 1.2. ■

It follows clearly from the preceding proposition that the Substitution Theorem remains valid for complex formulae that may feature one or both of the two unary operations $\neg; \#$.

Proposition 7.2 $\neg \frac{3}{4} \mathbb{R} \equiv \mathbb{R} \wedge \mathbb{R} \frac{1}{2} \mathbb{R} ? \wedge \mathbb{R} \frac{3}{4} \# \neg$;

Proof. Use (Adjointness) and P13. ■

Theorem 7.1 $\neg : \mathcal{A} \rightarrow \mathcal{B} \Rightarrow \neg : \mathcal{B} \rightarrow \mathcal{A}$ and $\neg : \mathcal{A} \rightarrow \mathcal{B} \Rightarrow \neg : \mathcal{B} \rightarrow \mathcal{A}$:

Proof. The first equivalidity follows from Theorem 6.14, and the second one from Theorem 6.4. The third one is a consequence of the first and the second. ■

We see from Proposition 7.1 and Theorem 7.1 that \neg is a negation function.

Theorem 7.2 $\neg \# \mathcal{A} \Rightarrow \neg : \mathcal{A} \rightarrow \mathcal{A}$:

Proof. Direct from Theorem 1.1 and Conclusion 1.1. ■

Theorem 7.3 $\neg : (\mathcal{A} \rightarrow \mathcal{B}) \Rightarrow \mathcal{A} \rightarrow \mathcal{B}$ and $\neg \# (\mathcal{A} \rightarrow \mathcal{B}) \Rightarrow \mathcal{A} \rightarrow \mathcal{B}$:

Proof. Direct from Theorems 6.6, 6.8. ■

Theorem 7.4 $\neg \# \mathcal{A} \Rightarrow \mathcal{A}$ and $\neg : \mathcal{A} \rightarrow \mathcal{A}$:

Proof. Direct from P2 and Theorem 1.2. ■

Theorem 7.5 $\neg \# \mathcal{A} \Rightarrow \mathcal{A}$ and $\neg \# \mathcal{A} \Rightarrow \mathcal{A}$:

Proof. Direct from Theorem 1.2 and P13. ■

Theorem 7.6 $\neg \# : \mathcal{A} \rightarrow \mathcal{A}$ and $\neg : \mathcal{A} \rightarrow \mathcal{A}$:

Proof. By Theorem 6 of [16], $((\neg \mathcal{A} \Rightarrow) \Rightarrow) \mathcal{A} \Rightarrow (\neg \mathcal{A} \Rightarrow)$, which is the first equivalidity. We get the second equivalidity from Theorem 7 of [16].

■

It follows from Theorem 7.6 that the two unary operations $\neg \#$ and \neg are idempotent, up to equivalidity.

Theorem 7.7 $\neg \# \mathcal{A} \Rightarrow (\neg \mathcal{A} \Rightarrow)$, $\neg : \mathcal{A} \rightarrow \mathcal{A} \Rightarrow \mathcal{A} \rightarrow \mathcal{A}$, $\neg \# \mathcal{A} \Rightarrow (\neg \mathcal{A} \Rightarrow)$ and $\neg \# \mathcal{A} \Rightarrow (\neg \mathcal{A} \Rightarrow)$:

Proof. Direct from Proposition 1.2 and P13. ■

Proposition 7.3 $\neg \# \mathcal{A} \Rightarrow \mathcal{A} \Rightarrow \neg \# \mathcal{A} \Rightarrow$ and $\neg \# \mathcal{A} \Rightarrow \mathcal{A} \Rightarrow \neg \# \mathcal{A} \Rightarrow$:

Proof. By Adjointness. ■

Proposition 7.4 $\neg : \mathcal{A} \rightarrow \mathcal{A} \Rightarrow \neg : \mathcal{A} \rightarrow \mathcal{A} \Rightarrow$:

Proof. (1) $\neg : \mathcal{A} \rightarrow \mathcal{A} \Rightarrow \neg$ (hypothesis, Substitution Theorem)
 (2) $\neg : \mathcal{A} \rightarrow \mathcal{A} \Rightarrow \neg$ (Theorem 7.6)
 (3) $\neg : \mathcal{A} \rightarrow \mathcal{A} \Rightarrow \neg$ (hypothesis, (1), (2)). ■

Proposition 7.5 $\neg \# \mathcal{A} \Rightarrow \neg \# \mathcal{A} \Rightarrow$:

Proof. Similar. ■

Proposition 7.6 $\# \neg \neg (\#) \neg \neg \#$ and $\# \neg \neg (\# \neg \neg) \neg \neg \#$:

Proof. Direct from Proposition 1.2. ■

With the help of the preceding proposition, it is easy to deduce the following Modus Tollens schema.

Proposition 7.7 $\# \neg \neg \#$; $\# \neg \neg \#$;

$\# \neg \neg \#$; $\# \neg \neg \#$;

$\# \neg \neg \#$; $\# \neg \neg \#$;

$\# \neg \neg \#$; $\# \neg \neg \#$;

$\# \neg \neg \#$; $\# \neg \neg \#$;

Conclusion 7.1 The following ...ve schema of formulae are equivalent in Adj LPC:

N1: $\# \neg \neg \#$;

N2: $\# \neg \neg \#$;

N3: $\# \neg \neg \#$;

N4: $\# \neg \neg \#$;

N5: $\# \neg \neg \#$;

Proof. The equivalences N1 \wedge N2 and N3 \wedge N4 follow from Proposition 7.3.

N2 entails N3: By Theorem 7.4, $\# \neg \neg \#$, and so by Proposition 7.1, $\# \neg \neg \#$. But by N2 with $\frac{\#}{\#}$, $\# \neg \neg \#$. Consequently, $\# \neg \neg \#$.

N4 entails N1: Similar.

Finally, N5 is the conjunction of N1 and N3. ■

It is clear from the above conclusion that n need not equal m (see the next example), and equality will hold if and only if for all x, $x \cdot n(x)$.

Example 7.1 Define a conjunction K on [0; 1] by:

$$K(x; y) = \begin{cases} 0; & 2x + y < 1 \\ \min\{x; y\}; & 2x + y > 1 \end{cases}$$

This K is an associative conjunction with two-sided identity, but it is neither commutative nor continuous. It is direct to see that its implication triple is completed as follows:

$$A(x; z) = \begin{cases} 1; & x \cdot z \\ \max\{1 - 2x; z\}; & x > z \end{cases}$$

$$H(y; z) = \begin{cases} 1; & y \cdot z \\ \max\{1 - y; z\}; & y > z \end{cases}$$

which are not comparable. For this adjointness lattice, we ...nd:

$$n(x) = A(x; 0) = \begin{cases} 1; & x = 0 \\ \max\{1 - 2x; 0\}; & x > 0 \end{cases}$$

$$m(y) = H(y; 0) = \begin{cases} 1; & y \cdot z \\ (1 - y); & y > z \end{cases}$$

So, $n \notin m$. Also, we note that each of the two inequalities $x \cdot n(x)$ and $x \cdot m(m(x))$ fails for some x.

8 S-type Implications

In this final section, we consider a type of implications that has seen sufficient interest in the literature. Given a strong negation n and a triangular norm T on $(L; \cdot)$, the S-type implication of T and n is defined on $(L; \cdot)$ by:

$$A(x; y) = n(T(x; n(y))); \quad (12)$$

For simplicity of terminology, we shall say that an adjointness lattice is of the S-type if so its implication A is. (N.B. It is direct to verify that if A is given by (12), then H will be the n -contrapositive of the residuated implication J_T of T , whereas K will be given by $K(x; y) = n(J_T(x; n(y)))$; $x; y \in L$.)

Our aim is to prove that adjoining to EP-AdjLPC one extra "involution" axiom (for the negation \neg) renders the implication \rightarrow an S-type implication. We denote the ensuing syntax by S-AdjLPC. Its language is that of AdjLPC. It is well known that in an adjointness lattice of the S-type, A satisfies EP and n is involutive (see [18]). Therefore, the following axioms and inference rules are sound for those lattices:

Axioms of S-AdjLPC:

- P1: $\circ \frac{3}{4} (\circ) \circ$;
P2: $\circ \frac{3}{4} ((\circ) \circ) \frac{3}{4} \circ$;
P3: $((\circ \frac{3}{4} \neg) \neg) \frac{3}{4} \circ \frac{3}{4} (\circ \frac{3}{4} \circ)$;
P6: $((\neg \frac{3}{4} \circ) \& \neg) \& \pm \frac{3}{4} \circ \& \pm$;
E5: $(\neg \frac{3}{4} (\circ) \circ) \frac{3}{4} (\circ \& \neg \frac{3}{4} \circ)$;
E7: $(\circ \& \neg \frac{3}{4} \circ) \frac{3}{4} (\neg \frac{3}{4} (\circ) \circ)$;
P8: $\circ \wedge \neg \frac{3}{4} \circ \neg$;
P9: $\circ \neg \circ \frac{3}{4} \circ$;
P10: $(\neg \frac{3}{4} \circ) \frac{3}{4} (\circ \neg \frac{3}{4} \circ \neg \circ)$;
P11: $\neg \frac{3}{4} \neg \wedge \neg$;
P12: $(\neg \frac{3}{4} \circ) \frac{3}{4} (\circ \wedge \neg \frac{3}{4} \circ \wedge \circ)$;
P13: $? \frac{3}{4} \circ$;
IN: $\circ \frac{1}{2} \frac{3}{4} : : \circ$.

Inference Rules of S-AdjLPC: MP, I2 and I3.

The remaining arguments of this section are carried out within S-AdjLPC, whereby inferencing from its axioms will be denoted by \vdash_s . Recall that S-AdjLPC is just EP-AdjLPC with the involution axiom IN added. Accordingly, we are entitled to use all theorems and established inferences of EP-AdjLPC.

In terms of \rightarrow and \neg , we define the following, new logical connective \boxtimes :

$$\circ \boxtimes \neg = : (\circ) : \neg); \quad (13)$$

We call it in S-AdjLPC the tie conjunction on WF.

Proposition 8.1 (monotonicity of \boxtimes).

$$\circ \frac{3}{4} \neg \vdash_s f \circ \boxtimes \circ \frac{3}{4} \neg \circ, \circ \boxtimes \circ \frac{3}{4} \circ \neg g:$$

Proof. Clear, from the monotonicity properties of \rightarrow and \neg . ■

Theorem 8.1 (commutivity of \boxtimes). $\vdash_S \textcircled{A} \boxtimes \neg \frac{1}{2} \frac{3}{4} \neg \textcircled{B} : \textcircled{B} \boxtimes \textcircled{A}$:

Proof. This is just the following equivalidity from EP and Substitution Theorem:

$$: (\textcircled{A}) (\neg \frac{3}{4} ?) \frac{1}{2} \frac{3}{4} : (\neg) (\textcircled{B} \frac{3}{4} ?) : \blacksquare$$

Theorem 8.2 (exchange principle for \boxtimes). $\vdash_S \textcircled{A} \boxtimes (\neg \textcircled{B} \circ) \frac{1}{2} \frac{3}{4} \neg \textcircled{B} (\textcircled{A} \circ) :$

Proof. $\textcircled{A} \boxtimes (\neg \textcircled{B} \circ) = : (\textcircled{A}) : : (\neg) : \circ) \frac{1}{2} \frac{3}{4} : (\textcircled{B}) (\neg) : \circ) \textcircled{A}$ (by (IN)) $\frac{1}{2} \frac{3}{4} : (\neg) (\textcircled{B}) : \circ) \textcircled{A}$ (by (EP)) $\frac{1}{2} \frac{3}{4} : (\neg) : : (\textcircled{B}) : \circ) \textcircled{A}$ (by (IN)) = $\neg \textcircled{B} (\textcircled{A} \circ) : \blacksquare$

Theorem 8.3 (associativity of \boxtimes). $\vdash_S \textcircled{A} \boxtimes (\neg \textcircled{B} \circ) \frac{1}{2} \frac{3}{4} (\textcircled{B} \circ \neg) \textcircled{A} \circ :$

Proof. This is a routine consequence of the preceding two theorems. \blacksquare

Theorem 8.4 (identity element). $\vdash_S \textcircled{A} \boxtimes > \frac{1}{2} \frac{3}{4} \textcircled{A}$ and $\vdash_S > \textcircled{A} \frac{1}{2} \frac{3}{4} \textcircled{A}$

Proof. $\textcircled{A} \boxtimes > = : (\textcircled{A}) : > \frac{1}{2} \frac{3}{4} : (\textcircled{A}) ? = : : \textcircled{A} \frac{1}{2} \frac{3}{4} \textcircled{A}$ (by (IN)).

Also, $> \textcircled{A} \textcircled{A} = : (>) : \textcircled{A} \frac{1}{2} \frac{3}{4} : : \textcircled{A} \frac{1}{2} \frac{3}{4} \textcircled{A}$; by (IN). \blacksquare

It follows from Proposition 8.1 and Theorems 8.1-8.4 that the tie conjunction \boxtimes is a triangular norm. Also we know that \neg is a negation, and so by IN, \neg is a strong negation.

Now, from IN we have

$$(\textcircled{A}) \neg \frac{1}{2} \frac{3}{4} : : (\textcircled{A}) : : \neg) = : (\textcircled{A} \boxtimes \neg);$$

that is, \boxtimes is the S-type implication of these \boxtimes and \neg . This completes the proof that S-Adj LPC is a sound and complete syntax for the semantical domain of all adjointness lattices of the S-type.

We next study some essential features of S-type implications. The next theorem states that they satisfy self-contraposition.

Theorem 8.5 $\vdash_S (\textcircled{A}) : \circ) \frac{1}{2} \frac{3}{4} (\circ) : \textcircled{A}$;

$$\vdash_S (\textcircled{A}) \circ) \frac{1}{2} \frac{3}{4} (: \circ) : \textcircled{A},$$

$$\vdash_S (\neg \frac{3}{4} : \textcircled{A}) \frac{1}{2} \frac{3}{4} : (\textcircled{A} \& \neg),$$

$$\vdash_S : \textcircled{A} \frac{1}{2} \frac{3}{4} \# \textcircled{A}.$$

Proof. The ...rst equivalidity holds by EP. The second equivalidity follows from the ...rst one and IN. The third equivalidity is just a restatement of axioms E5 and E7 with $\circ = ?$. The fourth holds by IN and Conclusion 7.1. \blacksquare

Proposition 8.2 $\neg \frac{3}{4} : \textcircled{A} \neg \textcircled{B} \frac{3}{4} : \neg$ and $\textcircled{A} \& \neg \frac{1}{2} \frac{3}{4} ? \neg \textcircled{B} \frac{1}{2} \frac{3}{4} ? :$

Proof. The ...rst equivalence follows from Proposition 7.2 and $\vdash_S : \textcircled{A} \frac{1}{2} \frac{3}{4} \# \textcircled{A}$ (Theorem 8.5). The second equivalence follows from the ...rst one and Proposition 7.2. \blacksquare

Proposition 8.3 $\textcircled{A} \frac{1}{2} \frac{3}{4} \neg \neg \textcircled{B} : \textcircled{B} \frac{1}{2} \frac{3}{4} : \neg :$

Proof. This follows directly from Proposition 7.1 and IN. ■

The next theorem justifies the terminology “tie conjunction” for \multimap . For a general study of such conjunctions in adjointness algebras, see [1].

Theorem 8.6

$$\multimap_S (\otimes) (\dashv) \circ) \frac{1}{2} \frac{3}{4} (\otimes \multimap \dashv) \circ); \quad (14)$$

$$\multimap_S \otimes \& (\dashv \& \circ) \frac{1}{2} \frac{3}{4} (\otimes \multimap \dashv) \& \circ; \quad (15)$$

Proof. We have the following equivalities; by IN and the associativity of \multimap :

$$\begin{aligned} (\otimes) (\dashv) \circ) \frac{1}{2} \frac{3}{4} : (\otimes \multimap : (\dashv) \circ) \frac{1}{2} \frac{3}{4} : (\otimes \multimap : : (\dashv \multimap : \circ) \frac{1}{2} \frac{3}{4} \\ : (\otimes \multimap (\dashv \multimap : \circ) \frac{1}{2} \frac{3}{4} : ((\otimes \multimap \dashv) \multimap : \circ) \frac{1}{2} \frac{3}{4} (\otimes \multimap \dashv) \circ). \end{aligned}$$

This proves (14).

By repeated application of (Adjointness), we obtain the following equivalences:

$$\otimes \& (\dashv \& \circ) \frac{3}{4} \pm \multimap_S \dashv \& \circ \frac{3}{4} (\otimes) \pm) \multimap_S \circ \frac{3}{4} (\dashv) (\otimes) \pm) \multimap_S \circ \frac{3}{4} (\dashv \multimap \otimes) \pm) \multimap_S (\dashv \multimap \otimes) \& \circ \frac{3}{4} \pm \text{ (by (14))} .$$

So by commutivity of \multimap (Theorem 8.1) we get the equivalence $\otimes \& (\dashv \& \circ) \frac{3}{4} \pm \multimap_S (\otimes \multimap \dashv) \& \circ \frac{3}{4} \pm$:

Now, (15) ensues from applying this equivalence twice; once with $\frac{\otimes \& (\dashv \& \circ)}{\pm}$, and again with $\frac{(\otimes \multimap \dashv) \& \circ}{\pm}$. ■

We next study the effects of adjoining to S-AdjLPC the following commutativity axiom for $\&$:

$$\text{COM: } \otimes \& \dashv \frac{3}{4} \dashv \& \otimes$$

Proposition 8.4

$$\text{COM } \multimap_S (\otimes) \circ) \frac{1}{2} \frac{3}{4} (\otimes \frac{3}{4} \circ); \quad (16)$$

$$\text{COM } \multimap_S \otimes \& \dashv \frac{1}{2} \frac{3}{4} \otimes \multimap \dashv; \quad (17)$$

Proof. We have the equivalences:

$$\dashv \frac{3}{4} (\otimes) \circ) \multimap_S \otimes \& \dashv \frac{3}{4} \circ \text{ (Adjointness) } \multimap_S \dashv \& \otimes \frac{3}{4} \circ \text{ (COM)}.$$

So by (Adjointness), $\dashv \frac{3}{4} (\otimes) \circ) \multimap_S \dashv \frac{3}{4} (\otimes \frac{3}{4} \circ)$.

By applying this last equivalence, once with $\frac{\otimes}{\pm}$ and again with $\frac{\otimes \frac{3}{4} \circ}{\pm}$, we get (16).

Next, assuming COM, we get the following equivalities in S-AdjLPC:

$$\begin{aligned} \otimes \multimap \dashv \frac{1}{2} \frac{3}{4} \dashv \multimap \otimes \text{ (Theorem 8.1) } = : (\dashv) : \otimes) \frac{1}{2} \frac{3}{4} : (\dashv \frac{3}{4} : \otimes) \text{ (by(16)) } \frac{1}{2} \frac{3}{4} \\ : : (\otimes \& \dashv) \text{ (Theorem 8.5) } \frac{1}{2} \frac{3}{4} \otimes \& \dashv \text{ (IN)}. \text{ This renders (17)}. \blacksquare \end{aligned}$$

The preceding proposition means that in S-AdjLPC enriched by COM, the implication \dashv is indistinguishable from the forcing implication $\frac{3}{4}$, and the conjunction $\&$ is indistinguishable from the triangular norm \multimap . Thus, COM

provides a complete characterization of an S-type implication \rightarrow , of some triangular norm \otimes , that is simultaneously the residuated implication of that \otimes .

We remark that in residuated logic we have another complete characterization of such implications. They are those residuated implications (of triangular norms) that satisfy IN. For an algebraic proof, see [18].

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